The Weibull Distribution
The Mathematics

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In economics we can use the Weibull distribution to model (1) random event arrival time and (2) random number of events realized within a given time interval. In this white paper we will develop the mathematics for the Weibull distribution. To that end we will work through the following hypothetical problem...

Our Hypothetical Problem

We are tasked with answering the following questions applicable to a Weibull distribution with shape parameter = 1.50 and scale parameter = 2.75.

Question 1: Graph the probability distribution where κ (shape) = 1.50 and λ (scale) = 2.75.
Question 2: What is the mean and variance of the distribution?
Question 3: What is the probability that the event will arrive after 3.00 years?
Question 4: Given that we pulled 0.35 from a uniform distribution what is the event arrival time?

The Weibull Distribution

We will define the variable t to be time in years and the function $f(t)$ to be the probability density function (PDF) of a Weibull-distributed random variable t. The equation for the probability density function where the variable κ is the shape parameter and the variable λ is the scale parameter is...

$$\text{PDF}: \quad f(t)=\frac{\kappa}{\lambda}\left(\frac{t}{\lambda}\right)^{\kappa-1}\text{Exp}\left\{-\left(\frac{t}{\lambda}\right)^{\kappa}\right\} \text{ ...where... } t\geq 0,\ \kappa>0,\ \lambda>0 \tag{1}$$

Note that when the shape parameter $\kappa < 1$ the probability density function $f(t)$ at $t = 0$ is undefined. Using Equation (1) above this statement in equation form is...

$$f(0)=\frac{\kappa}{\lambda}\left(\frac{0}{\lambda}\right)^{\kappa-1}\text{Exp}\left\{-\left(\frac{0}{\lambda}\right)^{\kappa}\right\}=\frac{\kappa}{\lambda}\,(0)^{\kappa-1}\ \Bigg|\ f(0)=0 \text{ ..when... } \kappa\geq 1\ \Bigg|\ f(0)=\infty \text{ ..when... } \kappa<1 \tag{2}$$

Note that when the shape parameter $\kappa = 1$ the Weibull distribution PDF becomes the Exponential distribution PDF. Using Equation (1) above this statement in equation form is... [1]

$$f(t)=\frac{1}{\lambda}\left(\frac{t}{\lambda}\right)^{1-1}\text{Exp}\left\{-\left(\frac{t}{\lambda}\right)^{1}\right\}=a\,\text{Exp}\left\{-a\,t\right\} \text{ ...where... } a=\frac{1}{\lambda} \tag{3}$$

Using Equation (1) above the equation for the cumulative probability that the Weibull-distributed random variable t is greater than or equal to a and less than or equal to b is...

$$\text{Prob}\Bigg[a\leq t\leq b\Bigg]=\int\limits_a^b f(t)\,\delta t=\int\limits_a^b\frac{\kappa}{\lambda}\left(\frac{t}{\lambda}\right)^{\kappa-1}\text{Exp}\left\{-\left(\frac{t}{\lambda}\right)^{\kappa}\right\}\delta t \tag{4}$$

Using Appendix Equation (23) below the solution to Equation (4) above is...

$$\text{Prob}\Bigg[a\leq t\leq b\Bigg]=\text{Exp}\left\{-\left(\frac{a}{\lambda}\right)^{k}\right\}-\text{Exp}\left\{-\left(\frac{b}{\lambda}\right)^{k}\right\} \tag{5}$$

The cumulative distribution function (CDF) is the probability that the random variable t drawn from a Weibull distribution will take a value less than or equal to z. Using Appendix Equation (24) below the equation for the cumulative distribution function is...

$$\text{CDF}: \ \ \text{Prob}\Big[t \leq z\Big] = \int\limits_0^z f(t)\,\delta t = 1 - \text{Exp}\left\{-\left(\frac{z}{\lambda}\right)^k\right\} \ \text{...where...} \ \ t \geq 0, \ \ \kappa > 0, \ \ \lambda > 0 \tag{6}$$

Distribution Mean and Variance

Using Equation (1) above the equation for the first moment (FM)of the Weibull-distributed random variable x is...

$$FM = \int\limits_0^\infty t\,f(t)\,\delta t = \int\limits_0^\infty t\,\frac{\kappa}{\lambda}\left(\frac{t}{\lambda}\right)^{\kappa-1}\text{Exp}\left\{-\left(\frac{t}{\lambda}\right)^\kappa\right\}\delta t \tag{7}$$

Using Appendix Equations (30) and (31) below the solution to Equation (7) above is...

$$FM = \lambda\,\Gamma(1+\kappa^{-1}) = \lambda\,\text{EXP}(\text{GAMMALN}(1+\kappa^{-1})) \tag{8}$$

Using Equation (1) above the equation for the second moment (SM) of the Weibull-distributed random variable x is...

$$SM = \int\limits_0^\infty t^2\,f(t)\,\delta t = \int\limits_0^\infty t^2\,\frac{\kappa}{\lambda}\left(\frac{t}{\lambda}\right)^{\kappa-1}\text{Exp}\left\{-\left(\frac{t}{\lambda}\right)^\kappa\right\}\delta t \tag{9}$$

Using Appendix Equations (35) and (36) below the solution to Equation (9) above is...

$$SM = \lambda^2\,\Gamma(1+2\,\kappa^{-1}) = \lambda^2\,\text{EXP}(\text{GAMMALN}(1+2\,\kappa^{-1})) \tag{10}$$

Using Equation (8) above the equation for the mean of a Weibull-distributed random variable t is...

$$\text{Mean of random variable } t = FM = \lambda\,\Gamma(1+\kappa^{-1}) \tag{11}$$

Using Equations (8) and (10) above the equation for the variance of a Weibull-distributed random variable t is...

$$\text{Variance of random variable } t = SM - FM^2 = \lambda^2\left[\Gamma(1+2\,\kappa^{-1}) - \left(\Gamma(1+\kappa^{-1})\right)^2\right] \tag{12}$$

Monte Carlo Simulation

We will define the variable p to be the probability of a given event. Since the value of p lies in the range $[0,1]$ we can pull the random value of p from a uniform distribution with domain $[0,1]$. This statement in equation form is...

$$p = \text{Random number (i.e. cumulative probability) drawn from a uniform distribution with domain } [0,1] \tag{13}$$

We will define the variable z to be a random variable drawn from a Weibull distribution. Using Equation (6) above we can set the value of p from Equation (13) above to the following equality...

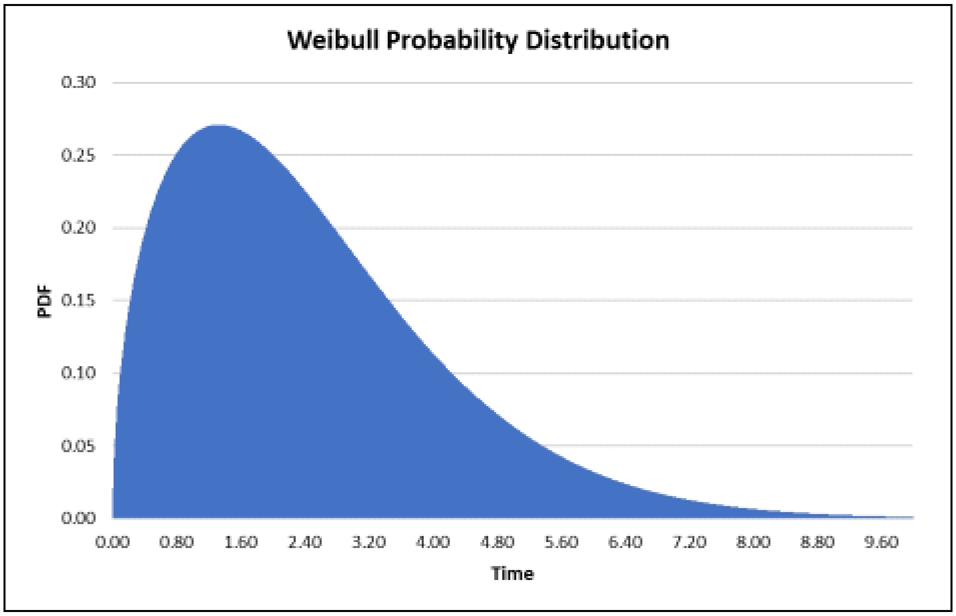
$$p = 1 - \text{Exp}\left\{-\left(\frac{z}{\lambda}\right)^k\right\} \tag{14}$$

Solving Equation (14) above for z we get the Weibull-distributed random variable z that is equivalent to the random variable p drawn from a uniform distribution...

$$z = \lambda\left[-\ln\left(1-p\right)\right]^{\frac{1}{k}} \tag{15}$$

The Answers To Our Hypothetical Problem

Question 1: Graph the probability distribution where κ (shape) = 1.50 and λ (scale) = 2.75.



Question 2: What is the mean and variance of the distribution?

Using Equations (16) above and (36) below the mean of the distribution is...

$$\text{Mean} = 2.75 \times \Gamma(1 + 1.50^{-1}) = 2.4825 \tag{16}$$

Using Equations (12) above and (36) below the mean of the distribution is...

$$\text{Variance} = 2.75^2 \times \left[\Gamma(1 + 2 \times 1.50^{-1}) - \left(\Gamma(1 + 1.50^{-1})\right)^2\right] = 2.8412 \tag{17}$$

Question 3: What is the probability that the event will arrive after 3.00 years?

Using Equation (6) above the answer to the question is...

$$\text{Prob}\left[t \geq 3.00\right] = 1 - \left(1 - \text{Exp}\left\{-\left(\frac{3.00}{2.75}\right)^{1.50}\right\}\right) = 1 - 0.68 = 0.32 \tag{18}$$

Question 4: Given that we pulled 0.35 from a uniform distribution what is the event arrival time?

Using Equation (15) above the answer to the question is...

$$z = 2.75 \times \left[-\ln\left(1 - 0.35\right)\right]^{\frac{1}{1.50}} = 1.57\,\text{years} \tag{19}$$

References

[1] Gary Schurman, *The Exponential Distribution,* March, 2012.

[2] Gary Schurman, *The Gamma Function,* April, 2016.

Appendix

A. We will define the functions $f(t)$ and $g(t)$ to be...

$$f(t) = \frac{\kappa}{\lambda}\left(\frac{t}{\lambda}\right)^{\kappa-1}\text{Exp}\left\{-\left(\frac{t}{\lambda}\right)^{\kappa}\right\} \text{ ...and... } g(t) = -\text{Exp}\left\{-\left(\frac{t}{\lambda}\right)^{k}\right\} \tag{20}$$

Using Equation (20) above the equation for the derivative of the function $g(t)$ is...

$$\frac{\delta}{\delta t}\,g(t)=-\left[-\kappa\left(\frac{t}{\lambda}\right)^{k-1}\times\frac{1}{\lambda}\times\text{Exp}\left\{-\left(\frac{t}{\lambda}\right)^{k}\right\}\right]=\frac{\kappa}{\lambda}\left(\frac{t}{\lambda}\right)^{\kappa-1}\text{Exp}\left\{-\left(\frac{t}{\lambda}\right)^{\kappa}\right\}=f(t) \tag{21}$$

Since the derivative of $g(t)=f(t)$ then the antiderivative of the function $f(t)$ is the function $g(t)$.

B. Using Equations (20) and (21) above the solution to the following integral is...

$$I=\int_a^b f(t)\,\delta t=\int_a^b \frac{\kappa}{\lambda}\left(\frac{t}{\lambda}\right)^{\kappa-1}\text{Exp}\left\{-\left(\frac{t}{\lambda}\right)^{\kappa}\right\}\delta t=-\text{Exp}\left\{-\left(\frac{t}{\lambda}\right)^{k}\right\}\Bigg[_{t=a}^{t=b} \tag{22}$$

The solution to Equation (22) above is...

$$I=-\text{Exp}\left\{-\left(\frac{b}{\lambda}\right)^{k}\right\}--\text{Exp}\left\{-\left(\frac{a}{\lambda}\right)^{k}\right\}=\text{Exp}\left\{-\left(\frac{a}{\lambda}\right)^{k}\right\}-\text{Exp}\left\{-\left(\frac{b}{\lambda}\right)^{k}\right\} \tag{23}$$

Note that when the lower and upper bounds of integration in Equation (22) above change to 0 and z, respectively, then Equation (23) above becomes...

$$I=\int_0^z f(x)\,\delta x=\text{Exp}\left\{-\left(\frac{0}{\lambda}\right)^{k}\right\}-\text{Exp}\left\{-\left(\frac{z}{\lambda}\right)^{k}\right\}=1-\text{Exp}\left\{-\left(\frac{z}{\lambda}\right)^{k}\right\} \tag{24}$$

Note that Equation (24) above is the cumulative distribution function for the Weibull-distributed random variable z.

C. The solution to fhe following integral is...

$$I=\int_0^\infty t\,f(t)\,\delta t=\int_0^\infty t\,\frac{\kappa}{\lambda}\left(\frac{t}{\lambda}\right)^{\kappa-1}\text{Exp}\left\{-\left(\frac{t}{\lambda}\right)^{\kappa}\right\}\delta t \tag{25}$$

We will define the variable u to be...

$$u=\left(\frac{t}{\lambda}\right)^{\kappa}\ \text{...such that...}\ t=\lambda\,u^{\frac{1}{\kappa}}\ \text{...and...}\ t^2=\lambda^2\,u^{\frac{2}{\kappa}} \tag{26}$$

The derivative of the variable u with respect to the random variable t is...

$$\frac{\delta u}{\delta t}=\frac{\kappa}{\lambda}\left(\frac{t}{\lambda}\right)^{\kappa-1}\ \text{...such that...}\ \delta u=\frac{\kappa}{\lambda}\left(\frac{t}{\lambda}\right)^{\kappa-1}\delta t \tag{27}$$

Using Equations (26) and (27) above we can rewrite Equation (25) above as...

$$I=\lambda\int_0^\infty u^{\frac{1}{\kappa}}\,\text{Exp}\left\{-u\right\}\delta u \tag{28}$$

Note that we can rewrite Equation (28) above as...

$$I=\lambda\int_0^\infty u^{\alpha-1}\,\text{Exp}\left\{-u\right\}\delta u\ \text{...where...}\ \alpha=1+\frac{1}{\kappa} \tag{29}$$

Noting that the integral in Equation (29) above is the Gamma Function, the solution to that integral is... [2]

$$I=\lambda\,\Gamma(\alpha)=\lambda\,\Gamma(1+\kappa^{-1}) \tag{30}$$

In Excel the gamma function of α is represented by the following Excel function... [2]

$$\Gamma(\alpha)=\text{EXP(GAMMALN}(\alpha))\ \text{...where...}\ \alpha>0 \tag{31}$$

D. The solution to fhe following integral is...

$$I = \int_{0}^{\infty} t^2 \, f(t) \, \delta t = \int_{0}^{\infty} t^2 \, \frac{\kappa}{\lambda} \left(\frac{t}{\lambda}\right)^{\kappa - 1} \text{Exp}\left\{ - \left(\frac{t}{\lambda}\right)^{\kappa} \right\} \delta t \tag{32}$$

Using Equations (26) and (27) above we can rewrite Equation (32) above as...

$$I = \lambda^2 \int_{0}^{\infty} u^{\frac{2}{\kappa}} \, \text{Exp}\left\{ - u \right\} \delta u \tag{33}$$

Note that we can rewrite Equation (33) above as...

$$I = \lambda^2 \int_{0}^{\infty} u^{\alpha - 1} \, \text{Exp}\left\{ - u \right\} \delta u \;\; \text{...where...} \;\; \alpha = 1 + \frac{2}{\kappa} \tag{34}$$

Noting that the integral in Equation (34) above is the Gamma Function, the solution to that integral is... [2]

$$I = \lambda^2 \, \Gamma(\alpha) = \lambda^2 \, \Gamma(1 + 2 \, \kappa^{-1}) \tag{35}$$

In Excel the gamma function of α is represented by the following Excel function... [2]

$$\Gamma(\alpha) = \text{EXP(GAMMALN}(\alpha)) \;\; \text{...where...} \;\; \alpha > 0 \tag{36}$$

E. Given the cumulative probability p and solving for the random variable z...

$$
\begin{aligned}
p &= 1 - \text{Exp}\left\{ - \left(\frac{z}{\lambda}\right)^{k} \right\} \\
\text{Exp}\left\{ - \left(\frac{z}{\lambda}\right)^{k} \right\} &= 1 - p \\
-\left(\frac{z}{\lambda}\right)^{k} &= \ln\left(1 - p\right) \\
\left(\frac{z}{\lambda}\right)^{k} &= -\ln\left(1 - p\right) \\
\frac{z}{\lambda} &= \left[- \ln\left(1 - p\right) \right]^{\frac{1}{k}} \\
z &= \lambda \left[- \ln\left(1 - p\right) \right]^{\frac{1}{k}} \qquad (37)
\end{aligned}
$$